Differentiability

Recall: If $f: S \longrightarrow \mathbb{R}$, $a \in S$

we say $\lim\limits_{x \to a} f(x) = c$

if $\lim\limits_{n \to \infty} f(x_n) = c$

for any sequence $(x_n) \longrightarrow a$

Definition: Let $f: I \to \mathbb{R}$, I some interval

$a \in I$

f is differentiable at a if $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists.

In this case, the limit is denoted by $f'(a)$.

<u>Examples:</u> 1 $f(x) = x^2$ $a=3$

check definition of differentiability:

$$\lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{x^2-3^2}{x-3}$$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}$$

$$= \lim_{x\to 3} x+3 = \boxed{6}$$

same procedure works for general a:

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{(x-a)}} = \lim_{x\to a} x+a = 2a$$

$\Rightarrow \boxed{f'(a) = 2a}$ can write $\boxed{f'(x) = 2x}$

② $f(x) = x^n$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} =$$

aside: $(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} \dots + a^{n-2}x + a^{n-1})$

$$= x^n + \cancel{ax^{n-1}} + \cancel{a^2x^{n-2}} + \dots + \cancel{a^{n-2}x^2} + \cancel{a^{n-1}x}$$
$$\quad - \cancel{ax^{n-1}} - \cancel{a^2x^{n-2}} - \dots - \cancel{a^{n-1}x} - a^n$$

$$= x^n - a^n$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x^{n-1} + ax^{n-2} + \dots + a^{n-1})}{\cancel{(x-a)}} =$$

$$= \lim_{x \to a} \underset{\substack{\downarrow \\ a^{n-1}}}{x^{n-1}} + \underset{\substack{\downarrow \\ a^{n-1}}}{ax^{n-2}} + \dots + \underset{\substack{\downarrow \\ a^{n-1}}}{a^{n-1}} = \boxed{na^{n-1}}$$

<u>Result:</u> $f'(a) = n a^{n-1}$

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$f(x) = \sqrt{x}$

$$f'(a) = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

(hint: factorize $x - a$!)

$= \ldots$

$= \frac{1}{2\sqrt{a}}$

(homework)

Properties:

Theorem: If f is differentiable at a
$\Rightarrow$ it is continuous at a

Proof. need to show: $\lim_{x\to a} f(x) = f(a)$

given: $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.

idea: try to use this limit to calculate that limit.

$$\lim_{x\to a} f(x) = \lim_{x\to a} \underbrace{\underbrace{(x-a)}_{\to 0} \cdot \underbrace{\frac{f(x)-f(a)}{x-a}}_{\to f'(a)}}_{=0} + f(a) = 0 + f(a)$$

$$= f(a)$$

✓

Properties of derivatives

Theorem f, g differentiable at a

$\Rightarrow$ ⓐ $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$ (product rule)

ⓑ $\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - g'(a)f(a)}{g(a)^2}$ If $g(a) \neq 0$

(quotient rule)

Proof. ⓐ need to calculate

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} =$$

$$= \lim_{x \to a} \frac{\left(f(x)g(x) - f(a)g(x)\right) + \left(f(a)g(x) - f(a)g(a)\right)}{x-a} =$$

$$= \lim_{x \to a} \underbrace{\frac{f(x)-f(a)}{x-a}}_{f'(a)} \underbrace{g(x)}_{g(a)} + \underbrace{f(a)}_{f(a)} \underbrace{\frac{g(x)-g(a)}{x-a}}_{g'(a)} =$$

$$= f'(a)\,g(a) + f(a)\,g'(a).$$

(b) $$\lim_{x \to a} \frac{f(x)/g(x) - f(a)/g(a)}{x-a} =$$

$$= \lim_{x \to a} \frac{f(x)\,g(a) - f(a)\,g(x)}{(x-a)\;g(x)\,g(a)}$$

factor $g(a)$ factor $f(a)$

$$= \lim_{x \to a} \frac{1}{g(x)\,g(a)} \frac{\big(f(x)\,g(a) - f(a)\,g(a)\big) + \big(f(a)\,g(a) - f(a)\,g(x)\big)}{x-a}$$

$$\lim_{x \to a} \frac{1}{g(x)g(a)} \left[\frac{f(x)-f(a)}{x-a}\, g(a) + \frac{g(a)-g(x)}{x-a}\, f(a) \right]$$

$$\frac{1}{g(a)^2} \qquad f'(a)\,g(a) \qquad -g'(a) \quad f(a)$$

$$= \frac{g(a)f'(a) - g'(a)f(a)}{g(a)^2}$$

Theorem (Chain Rule):

$f: S \to \mathbb{R}$ $\quad\quad$ $g: T \to \mathbb{R}$

assume $a \in S$ such we can find open interval $a \in I \subset S$

satisfying $f(I) \subset T$

$\Rightarrow$ $(g \circ f)'(a) = g'(f(a))\, f'(a)$

Proof. Case 1 $\quad$ Assume $f(x) \neq f(a)$ for x near a
$x \neq a$
(say for x in sufficiently small I)

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = \lim_{x \to a} \frac{[g(f(x)) - g(f(a))]\,(f(x) - f(a))}{\underbrace{(f(x) - f(a))}_{g'(f(a))}\ \underbrace{(x - a)}_{f'(a)}}$$